SOLUTION OF SOME PROBLEMS OF THE THEORY OF CREEP BY THE SMALL PARAMETER METHOD

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The small parameter method has been widely used recently to solve a number of complex elastic-plastic problems. In this method one starts from a known analytic solution of simpler (plane, axisymmetric, centrally symmetric) problems, and by eliminating some small quantities seeks solutions close to these known states. Many such problems are presented in [1]. However, this method has not been widely used in the theory of creep, because of the lack of exact analytic solutions of even the simplest problems. An exception is the case of steady creep, when elastic deformations are negligible, and the creep flow rates are determined only by the stress state, and do not depend on the loading history [2]. Solving problems by this scheme is equivalent to using the equations of physically nonlinear elasticity. Some of these problems were also treated in [1]. Taking account of elastic deformations and the loading history leads to such complications that analytic solutions cannot be obtained. Therefore, such problems are solved by well-known numerical methods (finite differences, finite elements). However, the use of these methods involves considerable difficulties in the debugging of programs, and a large volume of computations. On the other hand, these problems can be solved by the small parameter method involving very simple numerical procedures on a computer (e.g., the evaluation of certain integrals).

As an example, let us consider the problem of the deformation of an initially unstressed cylindrical tube with inside and outside radii R_1 and R_2 for plane strain and the following boundary conditions:

$$\sigma_r(R_1) = \sigma_{r\theta}(R_1) = 0,\ \ \sigma_r(R_2) = P(1 - \delta\cos 2\theta),\ \ \sigma_{r\theta}(R_2) = P\delta\sin 2\theta,$$

where σ_r and $\sigma_{r\theta}$ are the radial and tangential stresses, P and δ are constants ($0 < \delta < 1$), and θ is the angle in a polar coordinate system. We note that as $R_2 \to \infty$ these conditions correspond to the extension of an infinite plane with a circular hole, which is free of loads, by two perpendicular forces applied at infinity [1]. We assume that the material of the tube is isotropic, viscoelastic, and incompressible with respect to both elastic and viscous strains, and we write the strain tensor as the sum of the elastic strain and creep flow tensors. As a result we obtain [1]

$$\varepsilon_\theta = -\varepsilon_r = \frac{3}{4}\frac{u}{E} + \varepsilon_\theta^c,\ \ \varepsilon_{r\theta} = \frac{3}{4}\frac{v}{E} + \varepsilon_{r\theta}^c, \tag{1}$$

where $u = \sigma_\theta - \sigma_r$; $v = 2\sigma_{r\theta}$; σ_θ is the circumferential stress, ε_θ^c and $\varepsilon_{r\theta}^c$ are the circumferential and tangential creep flow components, and E is Young's modulus. For simplicity, we assume that the components of the strain rate are determined solely by the stress components, and are power functions of them [1, 2]:

$$\eta_\theta = \dot{\varepsilon}_\theta^c = \frac{3}{4}B(u^2 + v^2)^{\frac{n-1}{2}}u,\ \ \eta_{r\theta} = \dot{\varepsilon}_{r\theta}^c = \frac{3}{4}B(u^2 + v^2)^{\frac{n-1}{2}}v, \tag{2}$$

where B and n are creep constants.

We choose δ as the small parameter. Henceforth, we refer all the stress components to P, and the strain components to (3/4)P/E, retaining their previous notation, and introduce the dimensionless radius r, referred to R_2.

We consider the zero approximation, corresponding to the axisymmetric state of the tube for $\delta = 0$. We assume that at time $t = 0$ the creep flows are zero. Clearly in this case $\sigma_{r\theta}^{(0)} = 0$. The computational procedure consists in the following [3]. We assume that at the time t_k the distribution of creep flow $\varepsilon_\theta^{(0)c}$ along the radius r is known (at $t = 0$, $\varepsilon_\theta^{(0)c} = 0$

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everywhere in the tube). It follows from the conditions of plane strain and incompressibility that $\varepsilon_\theta^{(0)} = u^{(0)} + \varepsilon_\theta^{(0)c} = \frac{A_1}{r^2}$, from which $u^{(0)} = \frac{A_1}{r^2} - \varepsilon_\theta^{(0)c}$, where A_1 = const. From the equation of equilibrium $\sigma_r^{(0)\prime} = u^{(0)}/r$ (from now on a prime denotes differentiation with respect to the coordinate r) we obtain

$$\sigma_r^{(0)} = -\frac{A_1}{2r^2} - \int_\alpha^r \frac{\varepsilon_\theta^{(0)c}}{r}\, dr + A_2, \tag{3}$$

where $\alpha = R_1/R_2$, from which

$$\sigma_\theta^{(0)} = u^{(0)} + \sigma_r^{(0)} = \frac{A_1}{2r^2} - \varepsilon_\theta^{(0)c} - \int_\alpha^r \frac{\varepsilon_\theta^{(0)c}}{r}\, dr + A_2. \tag{4}$$

The constants A_1 and A_2 are determined from the boundary conditions $\sigma_r^{(0)}(\alpha) = 0$, $\sigma_r^{(0)}(1) = 1$:

$$A_1 = \frac{2\alpha^2}{1-\alpha^2}\left(1 + \int_\alpha^1 \frac{\varepsilon_\theta^{(0)c}}{r}\, dr\right),\ A_2 = \frac{A_1}{2\alpha^2}. \tag{5}$$

Assuming that the state of stress (3)-(5) does not change during the time Δt, we find the distribution of the strain ε_θ^c at time $t_{k+1} = t_k + \Delta t : \varepsilon_\theta^{(0)c}(t_{k+1}) = \varepsilon_\theta^{(0)c}(t_k) + \dot{\varepsilon}_\theta^{(0)c}(t_k)\,\Delta t$, where $\dot{\varepsilon}_\theta^{(0)c}$ is found from (2). The computational procedure is repeated. Thus, at any time the distributions of $\sigma_r^{(0)}$, $\sigma_\theta^{(0)}$, and $\varepsilon_\theta^{(0)c}$ along the radius r will be known in the zero approximation. The required stresses are written as series in the parameter δ:

$$\sigma_r = \sigma_r^{(0)} + \delta\sigma_r^{(1)} + \delta^2\sigma_r^{(2)} + \ldots,$$
$$\sigma_\theta = \sigma_\theta^{(0)} + \delta\sigma_\theta^{(1)} + \delta^2\sigma_\theta^{(2)} + \ldots,\ \sigma_{r\theta} = \delta\sigma_{r\theta}^{(1)} + \delta^2\sigma_{r\theta}^{(2)} + \ldots \tag{6}$$

Then, on the basis of (1), (2), and (6), we obtain for the first approximation of the strains

$$\varepsilon_\theta^{(1)} = u_1^{(1)} + \varepsilon_\theta^{(1)c},\ \varepsilon_{r\theta}^{(1)} = v_1^{(1)} + \varepsilon_{r\theta}^{(1)c}, \tag{7}$$

where the creep flow rates are determined in the following way:

$$\dot{\varepsilon}_\theta^{(1)c} = Anu^{(0)n-1}u_1^{(1)},\ \dot{\varepsilon}_{r\theta}^{(1)c} = Au^{(0)n-1}v_1^{(1)},\ A = EBP^{n-1}. \tag{8}$$

In any approximation it is necessary to satisfy the equilibrium equations [1]

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0,\ \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} = 0 \tag{9}$$

and the equation of compatibility of strains, which in the case under consideration of an incompressible medium under plane strain takes the form

$$r^2\frac{\partial^2\varepsilon_\theta}{\partial r^2} + 3r\frac{\partial\varepsilon_\theta}{\partial r} - \frac{\partial^2\varepsilon_\theta}{\partial\theta^2} = 2\left(\frac{\partial\varepsilon_{r\theta}}{\partial\theta} + r\frac{\partial^2\varepsilon_{r\theta}}{\partial r\partial\theta}\right). \tag{10}$$

Eliminating $(\sigma_r + \sigma_\theta)/2$ from (9), we have

$$r^2\frac{\partial^2 v}{\partial r^2} + 3r\frac{\partial v}{\partial r} - \frac{\partial^2 v}{\partial\theta^2} = -2\left(\frac{\partial u}{\partial\theta} + r\frac{\partial^2 u}{\partial r\partial\theta}\right). \tag{11}$$

We seek the solution in the form

$$\sigma_r^{(1)} = \sigma_{r1}^{(1)}(r, t)\cos 2\theta,\ \sigma_\theta^{(1)} = \sigma_{\theta1}^{(1)}\cos 2\theta,\ \sigma_{r\theta}^{(1)} = \sigma_{r\theta1}^{(1)}(r, t)\sin 2\theta. \tag{12}$$

Then, for the creep flows in the first approximation we obtain

$$\varepsilon_\theta^{(1)c} = \varepsilon_{\theta1}^{(1)c}(r, t)\cos 2\theta,\ \varepsilon_{r\theta}^{(1)c} = \varepsilon_{r\theta1}^{(1)c}(r, t)\sin 2\theta,$$
$$u_1^{(1)} = u^{(1)}\cos 2\theta,\ v_1^{(1)} = v^{(1)}\sin 2\theta. \tag{13}$$

Substituting (12) and (13) into (10) and (11), and taking account of (7), we find

$$r^2u^{(1)''} + 3ru^{(1)'} + 4u^{(1)} - 4v^{(1)} - 4rv^{(1)'} = f(r, t),$$
$$r^2v^{(1)''} + 3rv^{(1)'} + 4v^{(1)} - 4u^{(1)} - 4ru^{(1)'} = 0, \quad (14)$$

where $f(r,t) = 4\left(\varepsilon_{r\theta 1}^{(1)c} + r\varepsilon_{r\theta 1}^{(1)c'}\right) - r^2\varepsilon_{\theta 1}^{(1)c''} - 4\varepsilon_{\theta 1}^{(1)c} - 3r\varepsilon_{\theta}^{(1)c'}$.

We note that the quantities $u^{(1)}$ and $v^{(1)}$ appearing in (14) depend on r and t only.

We assume that at time t_k the distribution of the strains $\varepsilon_\theta^{(1)c}$ and $\varepsilon_{r\theta}^{(1)c}$ along the radius of the tube is known (at $t = 0$ $\varepsilon_\theta^{(1)c} = \varepsilon_{r\theta}^{(1)c} = 0$). Integrating system (14) and performing some simple but rather tedious transformations, we obtain

$$u^{(1)} = F^{(1)} - \Phi^{(1)} + 2c_1^{(1)} + 2c_2^{(1)}r^2 + 2c_3^{(1)}r^{-4} - 2c_4^{(1)}r^{-2},$$
$$v^{(1)} = \Phi^{(1)} + 2c_1^{(1)} + 2c_2^{(1)}r^2 - 2c_3^{(1)}r^{-4} + 2c_4^{(1)}r^{-2}, \quad (15)$$

where $F^{(1)}(r) = 4r^2\left[\int\limits_\alpha^r \frac{1}{r^3}\left(\int\limits_\alpha^r \frac{\varepsilon_{r\theta 1}^{(1)c} - \varepsilon_{\theta 1}^{(1)c}}{r}\,dr\right)dr + \int\limits_\alpha^r \frac{\varepsilon_{r\theta 1}^{(1)c} - \varepsilon_{\theta 1}^{(1)c}}{r^3}\,dr\right] - \varepsilon_{\theta 1}^{(1)c}$;

$$\Phi^{(1)}(r) = \frac{4}{r^2}\left[\int\limits_\alpha^r rF(r)\,dr - 3\int\limits_\alpha^r\left(\frac{1}{r^3}\int\limits_\alpha^r r^3F(r)\,dr\right)dr\right].$$

The constants $c_1^{(1)}$, $c_2^{(1)}$, $c_3^{(1)}$ and $c_4^{(4)}$ are determined from the boundary conditions, which in the present case have the form

$$\sigma_{r1}^{(1)} = \Psi^{(1)} - c_1^{(1)} - c_3^{(1)}r^{-4} + 2c_4^{(1)}r^{-2},$$
$$\sigma_{\theta 1}^{(1)} = \sigma_{r1}^{(1)} + u^{(1)}, \quad \sigma_{r\theta 1}^{(1)} = \Phi^{(1)}/2 + c_1^{(1)} + c_2^{(1)}r^2 - c_3^{(1)}r^{-4} + c_4^{(1)}r^{-2}, \quad (16)$$

where $\Psi^{(1)}(r) = \int\limits_\alpha^r \frac{F^{(1)}(r) - 2\Phi^{(1)}(r)}{r}\,dr.$

$$\sigma_{r1}^{(1)}(\alpha) = \sigma_{r\theta 1}^{(1)}(\alpha) = 0, \; \sigma_{r1}^{(1)}(1) = -1, \; \sigma_{r\theta 1}^{(1)}(1) = 1.$$

Assuming that the stress state (15), (16) does not change during the time Δt, we find from (8) the distribution of creep flows at the time $t_{k+1} = t_k + \Delta t$: $\varepsilon_\theta^{(1)c}(t_{k+1}) = \varepsilon_\theta^{(1)c}(t_k) + \dot{\varepsilon}_\theta^{(1)c}(t_k)\,\Delta t$, $\varepsilon_{r\theta}^{(1)c}(t_{k+1}) = \varepsilon_{r\theta}^{(1)c}(t_k) + \dot{\varepsilon}_{r\theta}^{(1)c}(t_k)\,\Delta t$. The computational procedure is then repeated.

By using a method analogous to that described in [1], and making some simple transformations, the second approximation for the strains can be obtained in the form

$$\varepsilon_\theta^{(2)} = u^{(2)} + \varepsilon_\theta^{(2)c}, \quad \varepsilon_{r\theta}^{(2)} = v^{(2)} + \varepsilon_{r\theta}^{(2)c}, \quad (17)$$

where the creep flow rates are determined in the following way:

$$\dot{\varepsilon}_\theta^{(2)c} = Anu^{(0)n-2}u^{(2)} + \frac{A(n-1)}{4}u^{(0)n-2}\left[nu^{(1)2} + v^{(1)2}\right.$$
$$\left. + (nu^{(1)2} - v^{(1)2})\cos 4\theta\right], \quad \dot{\varepsilon}_{r\theta}^{(2)c} = \frac{A(n-1)}{2}u^{(0)n-2}u^{(1)}v^{(1)}\sin 4\theta. \quad (18)$$

Taking account of (17) and (18), we seek the solution for the second approximation in the form of a sum of axisymmetric and nonaxisymmetric states:

$$\sigma_r^{(2)} = \sigma_{r1}^{(2)}(r, t) + \sigma_{r11}^{(2)}(r, t)\cos 4\theta, \quad \sigma_\theta^{(2)} = \sigma_{\theta 1}^{(2)}(r, t) + \sigma_{\theta 11}^{(2)}(r, t)\cos 4\theta,$$
$$\sigma_{r\theta}^{(2)} = \sigma_{r\theta 1}^{(2)}(r, t)\sin 4\theta, \quad \varepsilon_\theta^{(2)} = \varepsilon_{\theta 1}^{(2)}(r, t) + \varepsilon_{\theta 11}^{(2)}(r, t)\cos 4\theta, \quad \varepsilon_{r\theta}^{(2)} = \varepsilon_{r\theta}^{(2)}(r, t)\sin 4\theta$$

for the following boundary conditions: $\sigma_{r1}^{(2)} = \sigma_{r11}^{(2)} = \sigma_{r\theta 1}^{(2)} = 0$ ($r = \alpha$ and $r = 1$). Skipping the tedious calculations, we present the final formulas for the second-approximation stresses analogous to (3)-(5) and (15), (16):

$$\sigma_{r1}^{(2)} = \frac{\alpha^2}{1-\alpha^2}\left(\frac{1}{\alpha^2} - \frac{1}{r^2}\right)\int\limits_\alpha^1 \frac{\varepsilon_{\theta 1}^{(2)c}}{r}\,dr - \int\limits_\alpha^r \frac{\varepsilon_{\theta 1}^{(2)c}}{r}\,dr,$$

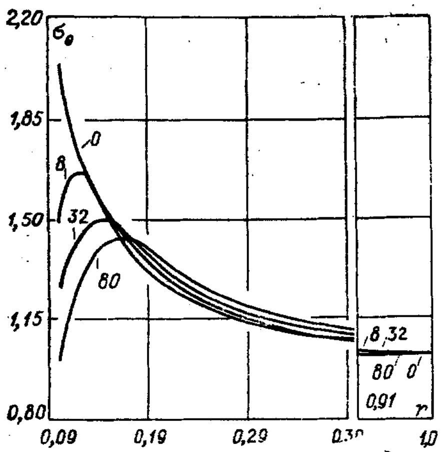

Fig. 1

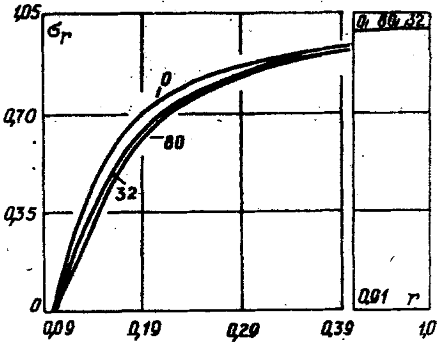

Fig. 2

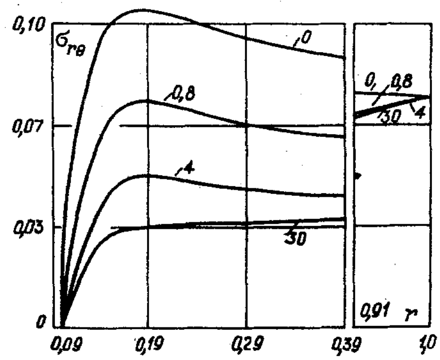

Fig. 3

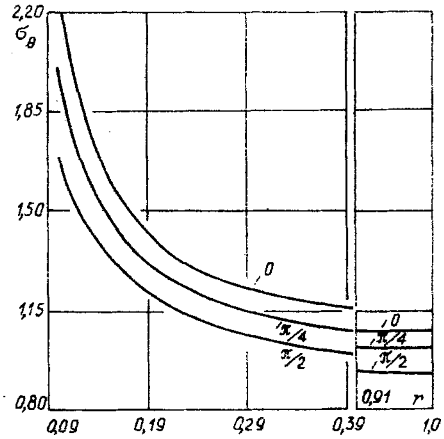

Fig. 4

$$\begin{aligned}
\sigma_{\theta 1}^{(2)} &= \frac{\alpha^2}{1-\alpha^2}\left(\frac{1}{\alpha^2}+\frac{1}{r^2}\right)\int_\alpha^1 \frac{\varepsilon_{\theta 1}^{(2)c}}{r}\,dr - \int_\alpha^r \frac{\varepsilon_{\theta 1}^{(2)c}}{r}\,dr - \varepsilon_{\theta 1}^{(2)c},\\
\sigma_{r11}^{(2)} &= \Psi^{(2)} + c_1^{(2)}r^2 - c_2^{(2)}r^4 - c_3^{(2)}r^{-6} + \frac{3}{2}c_2^{(2)}r^{-4},\\
\sigma_{\theta 11}^{(2)} &= F^{(2)} - \Phi^{(2)} + \Psi^{(2)} + c_1^{(2)}r^2 + \frac{3}{2}c_2^{(2)}r^4 + c_3^{(2)}r^{-6} - \frac{c_4^{(2)}}{2}r^{-4},\\
\sigma_{r\theta 1}^{(2)} &= \frac{1}{2}\Phi^{(2)} + c_1^{(2)}r^2 + c_2^{(2)}r^4 - c_3^{(2)}r^{-6} + c_4^{(2)}r^{-4},
\end{aligned} \qquad (19)$$

where

$$F^{(2)}(r) = 8r^4\left[3\int_\alpha^r \frac{1}{r^3}\left(\int_\alpha^r \frac{\varepsilon_{r\theta 1}^{(2)c}-\varepsilon_{\theta 11}^{(2)c}}{r^3}\,dr\right)dr + \int_\alpha^r \frac{\varepsilon_{r\theta 1}^{(2)c}-\varepsilon_{\theta 11}^{(2)c}}{r^5}\,dr\right] - \varepsilon_{\theta 11}^{(2)c};$$

$$\Phi^{(2)}(r) = -\frac{8}{r^4}\left[5\int_\alpha^r \frac{1}{r^3}\left(\int_\alpha^r r^5 F^{(2)}(r)\,dr\right) - \int_\alpha^r r^3 F^{(2)}(r)\,dr\right]; \quad \Psi^{(2)}(r) = \int_\alpha^r \frac{F^{(2)}(r) - 3\Phi^{(2)}(r)}{r}\,dr.$$

The constants $c_2^{(2)}$, $c_2^{(2)}$, $c_3^{(2)}$, and $c_4^{(2)}$ are found from the boundary conditions indicated. Subsequent approximations can be obtained in a similar way.

The following values of the constants were assumed in the calculation: $A = EBP^{n-1} = 0.827\ h^{-1}$, $n = 7$, $\alpha = 0.1$, $\delta = 0.08$, $\Delta t = 0.08$ h. All the integrals appearing in (3)-(5), (16), (15), and (19) were evaluated by using Simpson's rule. The number of division points along the radius set equal to 100, i.e., the integration step $\Delta r = 0.009$. The results of the calculation are shown in Figs. 1-5.

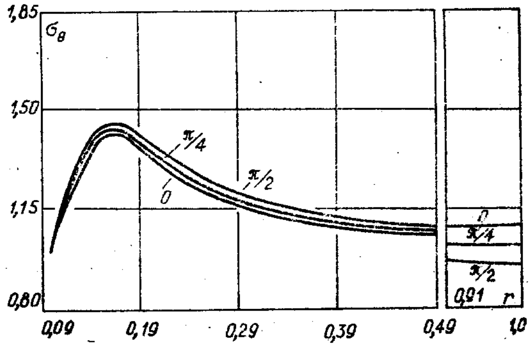

Fig. 5

Thus, Figs. 1-3 show the stresses σ_θ, σ_r, and $\sigma_{r\theta}$ at the various times indicated by the numbers on the curves for $\theta = \pi/4$. Figures 4 and 5 show the stress σ_θ at times 0 and 80 h for $\theta = 0$, $\pi/4$, and $\pi/2$. It can be seen from Figs. 1-3 that the most intense redistributions of the stresses σ_θ and $\sigma_{r\theta}$ occur at early times (0-32 h) near the inside opening of the tube. It can be seen from Figs. 4 and 5 that in the course of time the redistributions of the stress σ_θ for various values of the angle θ approach one another.

Calculation shows that taking account of the second approximation leaves the stress distribution and redistribution patterns practically unchanged; the maximum relative error in the stresses between the first and second approximations does not exceed 0.2%.

In the example considered above a quantity characterizing the perturbation of the boundary conditions in stresses was taken as the small parameter. It is clearly possible to consider the perturbation of geometrical boundary conditions and also both simultaneously. This permits the solution of a rather large class of problems analogous to those treated in [1] in the elastic–plastic formulation. Solution (3)-(5) must be taken as a basis for the stresses corresponding to an axisymmetric state.

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